Prelim Game Theory Cheat Sheet

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Normal Form Games

Players move simultaneously.
A normal form game $G = \{N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\}$ consists of:

- a finite set of players $N = \{1, ..., n\}$
- a finite set of strategies S_i for each player
- a Von Neumann-Morgenstern utility function $u_i : S \implies \mathbb{R}$ for each player, where $S = \times_{i\in N} S_i$ (Cartesian product) is the set of pure strategy profiles for all players.
  A particular pure strategy profile is denoted $s = (s_1, ...s_n)$

Normal form game explanation

- Von Neumann-Morgenstern utility function: Preference satisfy weak order, continuity and independence iff ∃ vNM utility function: $u : Z\mathbb{R}$ s.t. $x \succcurlyeq y \Leftrightarrow \sum_{z\in Z} u(z)x(z) \geq \sum_{z\in Z} u(z)y(z)$.
- Cartesian product:
  The Cartesuan product of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs where the elements of A are first and the elements of B are second.

Normal form game - mixed strategy

Mixed strategies are denoted $\sigma_i \in \Delta S_i$, and are elements of the set of probability distributions over S_i.
A mixed strategy profile for all players is denoted $\sigma = (\sigma_1, ..., \sigma_n) \in \times_{i\in N} \delta S_i = \Sigma$
Jistifications:

- It may be desirable to be unpredictable, and one can really think of mixed strategies as randomized actions. (need an example for desired unpredictable situation)
- Treat player i's strategy as other player's expectation of player i's strategy.

Payoffs: The payoff from a mixed strategy proflie is the expectation over utility from the pure strategy profiles under this probability distribution.

Weakly dominated strategies

Definition [Weakly dominated]: A strategy $\sigma_i \in \Delta S_i$ is *weakly dominated* by σ'_i
if $u_i(\sigma'_i, s_{-i}) \geq u_i(\sigma_i, s_{-i})$ for all $s_{-i} \in \Sigma_{-i}$
and $u_i(\sigma'_i, s_{-i}) > u_i(\sigma_i, s_{-i})$ for some $s_{-i} \in \Sigma_{-i}$.

A strategy σ_i is weakly dominated by σ'_i if σ'_i always gives at least as high a payoff, and for some strategy by the opponents' gives a strictly higher payoff.

Definition [Weakly dominant]: Strategy $s_i \in S_i$ is weakly dominant if it weakly dominates all other strategies. In other words, if all other strategies are weakly dominated by it.

⋆ **There is a tension between ruling out some strategies entirely and expecting them all to be played with positive probability, and so we see that order of removal can matter in Iterated Weakly Dominance, unlike Iterated Strict Dominance.**

Nash Equilibrium - Normal Form

Conditions	maximization payoffs; having correct beliefs
Description	Strategy profile $\sigma \in \Sigma$ is a NE if $\sigma_i \in B_i(\sigma_{-i}) \quad \forall i \in N$
Best Response	$B_i = argmax_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})$. Therefore each player is playing a best response to the choices made by the others.
Justification	under the NE, none of the players can strictly increase his payoff by an unilateral deviation.
Mixed NE	A mixed strategy is a best response, iff everything in its support is a best response; If two or more pure strategies are best responses, then any randomization over them is also a best response; Each player chooses probabilities so as to make her opponent indifferent.

Nash Equilibrium vs Dominance

- **Remark 1:** A dominated strategy cannot be a best response. A dominant strategy is the unique best response to all strategy profiles of others.
  Is there a dominant strategy for mixed strategy? - Yes. We have seen mixed dominant strategies, but never seen mixed dominated strategy.
- **Remark 2:** All NE survive IESDS
- **Remark 3:** If there is a unique strategy profile s^* that survives IESDS, then s^* is a NE

unique survivor profile from IESDS ⇔NE

Solutions to Normal form games

- If s_i is dominated by s_{-i} then it also will be dominated by any mixed strategies σ_{-i} that contain those pure strategies.
- A strategy can be dominated by mixed strategy.
- If player i's pure strategy s_i is strictly dominated (SDd) then so is any mixed strategy σ_i with s_i in its support.
- Even if a group of pure strategies are not strictly dominated, mixed strategies that combine them might be.

Solutions to Normal form games-IESDS

Procedure:

Iteratively remove strictly dominated pure strategies in any order.

When no more pure strategies can be removed, check the mixed strategies.

Solutions to Normal form games-After IESDS

Best Response:
Best response is based on all strategies after IESDS

- player i's mixed strategy is never a BR if there is a subset of opponents' strategies for which player i's particular strategy is not in the set of best response to those opponents' strategy.
- A best response correspondence for player 1 maps player 2's mixed strategies into player 1's mixed strategies. $B_1 : \Sigma_2 \implies \Sigma_1$. Construct correspondences by making pairwise comparisons of the payoffs from player 2's pure strategies.
- When a game has two players, the set of strictly dominated strategies σ_i is the same as those σ_i that are never a best response.
- When a game has three or more players, the. set of strictly dominated σ_i is smaller than the set of σ_i that are never a best response.

Nash Equilibrium - Interpretation (General)

- A NE is a minimal condition for self-enforcing behavior, because then players know what others will do and still don't want to deviate.
- NE could be the result of a pre-play agreement (outcomes are known at ex.ante)
- NE might be focal points, which appear to all players as the obvious correct choice of action.
- NE might be the result of learning or evolution, such that players converge to a NE after figuring out what others will do. *(Grim Trigger)*

Nash Equilibrium - Interpretation (Mixed)

- **Deliberate randomization:** players can guarantee themselves at least the reachable lower bound in expectation by randomizing between two actions.
- **Mixed equilibria represent averages of play over time:** we can interpret probability of playing a pure strategy as the proportion of games in which that pure strategy was played. Players want to play a best response to the maximum of the empirical distribution of strategies played in the past by her opponents.
- **Mixed equilibria represent population equilibria:** we don't know which type of person we will meet, but can form expectations based on percentages.
  (If 50% of the population plays tails, then the player who always plays heads is is happy to continue with that strategy.)
- **Purification:** players may not perfectly understand the game and its payoffs, and so might be slightly bias in their play towards heads or tails. If everyone believes that each player will obey his bias and play that strategy all of the time, then it's an equilibrium to always play you bias.

Extensive form games

Players can move at different times

- We usually assume that an extensive form game Γ exhibits *perfect recall*.
  a node cannot have a successor in its own information set.
  if node x and y are in the same information set of player i, then the choices made by i leading to x and y are identical.
- Each player's strategy must specify how she acts at each of her information sets h even when:
  other player's action prevented h form being reached.
  her own actions prevent h form being reached.

Extensive form games

- A *pure strategy* specifies an action of an information set: $s_i \in S_i \equiv \times_{h_i \in H_i} C_h$
- A *mixed strategy* is denoted $\sigma_i \in \Sigma_i \equiv \Delta \times_{h_i i} C_h$. Each element σ_i in $_i$ is a probability vector whose components (the probability of playing each pure strategy, which may include multiple information sets) sum to 1. The action at each information set is specified in advance of play.

[Different between an action and a strategy]: A strategy may contain multiple actions, one for each information set.

Extensive form games - mixed strategy

A pure strategy of player i assigns an action to each of player i's information sets.// *Mixed strategy* might be specified in such a way as to give us information about what a player would do at a different information set.

For example, for strategies (wy, wz, xy, xz), the mixed strategy $\hat{\sigma}_i = (\frac{1}{3}, \frac{1}{6}, 0 \frac{1}{2})$, we know that playing x at one information set means that the player would play z at the other information set (x and z are correlated).

However, this is not generally helpful, because if we've reached a particular information set, we can't go back once we know a player will act.

Concepts - Extensive

- **[Backwards induction:]** Ruling out the NE with "non-credible threat" is "backwards induction".
  The *generalized backwards induction* occurs when the information set is involved.
- **[Information set:]** Treat the information set as one action (by expectation).
- **[Subgame prefect:]** Backwards induction finds the SPNE.
- **[Zermelo's Theorem:]**

  Every finite (finite strategy space) game of perfect information has a pure strategy SPE
  If no player has the same payoff at any two terminal nodes and there are no moves by nature, then the SPE is unique

When specifying a SPNE (or any other equilibrium), I need to specify full equilibrium strategies. This include what the players were intending to do in subgames that are not reached along the path play (off-path)

Also the SPNE is not required to be reachable. But the NE must be accomplished.

Nash Equilibrium - Extensive

- Given an extensive form game Γ, beliefs are a map $\mu : D \to [0,1]$ s.t. $\sum_{x \in h} \mu(x) = 1$ for all information sets h.
- $\mu(x) = P_\sigma(x|h) = \frac{P_\sigma(x)}{P_\sigma(h)}$
  $P_\sigma(h) \Longrightarrow$ the probability of reaching information set h.
  $P_\sigma(x) \Longrightarrow$ the probability of reaching node $x \in h$. $P_\sigma(h) = \sum_{x \in h} P_\sigma(h)$
- A strategy profile is *sequentially rational* given beliefs μ for each player i and each information set $h \in H_i$, player i's behavior conditional on h being reached maximizes his expected utility given σ_{-i} and μ

Extensive form and SPE

Relationship between extensive form equilibrium refinements:
Extensive From perfect $\Longrightarrow$ Sequential $\Longrightarrow$ SPE or PBE $\Longrightarrow$ NE

In generic games (games with no ties between payoffs), extensive form perfect and sequential equilibria are the same.

Incomplete vs Imperfect information

- **[Prefect information:]**In a game of imperfect information,players are simply unaware of the actions chosen by other players. However they know who the other players are, what their possible strategies/actions are, and the preferences/payoffs of these other players. Hence, information about the other players in imperfect information is complete.no information set is in the game tree
- **[Incomplete information:]** In incomplete information games, players may or may not know some information about the other players, e.g. their "type", their strategies, payoffs or their preferences.

Behavior vs mixed strategies(minor)

- **[Mixed strategies:]** A mixed strategy is simply a probability distribution over a complete, contingent plan on actions. Each player randomly chooses a pure strategy plan at the beginning of the game, and follows that plan through out play.
- **[Behavior strategies:]** A behavior strategy assigns a randomized action to each information set of the player. It is a complete contingent plan of (possibly) randomization actions.

As I understand, the mixed strategies narrows down the behavior strategies. We can consider the mixed strategy as a path, but the behavior strategy is all possible actions that generated by the support.

Nash Equilibrium - Repeated games

- **[Unique NE]** If a stage game is repeated finitely many times ($T < \infty$), and if it has a unique NE, then the repeated game has a unique perfect equilibrium the outcome of which is the repetition of the unique NE of the stage game.
- **[Grim trigger]**
  $V_0 = \frac{a}{1-\delta}$;
  $V_1 = A + \frac{\delta b}{1-\delta}$
  Myopic gain: $A - a$

  Compare V_0 and V_1 to decide whether to deviate.
- **[Folk theorem]**

Nash Equilibrium - Signaling games

Important concepts (with 4 requirements):

- [**Belief:**] A belief is a probability distribution over the nodes in an information set.
- **R1** [**Belief system:**] A belief system assigns such a belief to each information set. In every information set, the player that this information set belongs to has a belief about where he is within it.
- **R2** [**Sequential rationality:**] Each player in each of his information sets acts optimally given his beliefs and the strategies of others. This is called sequential rationality.
  We could calculate the expected payoff under each action. Then we can pin down when we will choose a certain action as the p varies. When each states' expected payoff are equal, we randomize the choice among them.
- **R3** [**Typical beliefs:**] At information sets that are on the equilibrium path, beliefs are derived from the strategies via Bayes rule.
  In the signaling game, we use μ denote this Bayesian belief, I prefer to use the conditional probability.
- **R4** [**Off path equilibrium:**] At information sets off-the equilibrium path, beliefs must be derived from strategies using Bayes rule, whenever possible.

When equilibrium satisfying the first three requirements, it is called a Bayesian Equilibrium. When satisfying all four requirements, it is called Perfect Bayesian equilibrium.

Nash Equilibrium - Signaling games

Equilibrium refinements: Intuitive criterion

[**Equilibrium dominance**]**:** Fix a perfect Bayesian equilibrium. A message m_j is equilibrium dominated for a type t_i if type t_i's equilibrium payoff is greater than the highest possible payoff that type t_i can get when choosing m_j

Intuitive criterion check:

- m_j is an off equilibrium message
- m_j is equilibrium dominated for type t_i
- there exists a type $t_{i'}$ for whom m_j is not equilibrium dominated

then the beliefs should put zero probability on type t_i when m_j is observed;
i.e., $\mu(t_i|m_j) = 0$